

# Supplement for LM-cut and Operator Counting Heuristics for Numeric Planning with Simple Conditions: Counter Examples and Additional Proofs

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## Translating SCT into RT

Given a SCT  $\Pi^{\text{num}} = \langle \mathcal{F}_p, \mathcal{N}, \mathcal{A}, s_I, G \rangle$ , we define a transformed task  $\Pi^{\text{RT}} = \langle \mathcal{F}_p, \mathcal{N}^{\text{RT}}, \mathcal{A}^{\text{RT}}, s_I^{\text{RT}}, G^{\text{RT}} \rangle$  built as follows. For every numeric expression mentioned in every numeric condition  $\psi \in \bar{\mathcal{F}}_n$ , we add an additional numeric variable  $v^\psi \in \mathcal{N}^{\text{RT}}$ , with  $s_I[v^\psi] = \sum_{v \in \mathcal{N}} w_v^\psi s_I[v]$ . Each numeric condition is replaced by  $v^\psi \geq w_0^\psi$  and, for every action  $a$ , a numeric effect on every variable  $v^\psi$  must be added, with the form  $v^\psi \leftarrow v^\psi + \sum_{v \in \mathcal{N}} w_v^\psi k_v^a$ , where  $v \ += \ k_v^a \in \mathbb{Q}$  are the original numeric effects of the action. This translation is polynomial in the number of active numeric conditions of the planning task.

## Proofs

**Proposition 1.** *Given a planning task  $\Pi$ , cost partitioning  $\{\Pi_i\}_{i=1}^n$ , and an admissible heuristic function  $h_i$  for each  $\Pi_i$ , the heuristic function  $h(s) = \sum_{i=1}^n h_i(s)$  is admissible.*

*Proof.* Consider the case  $n = 2$ . For any state  $s$ , Let  $\pi$  be an optimal  $s$ -plan with the cost  $h^*(s)$  for  $\Pi$ . Since  $\Pi_1$  and  $\Pi_2$  are the same as  $\Pi$  except for the cost functions,  $\pi$  is also a valid  $s$ -plan for both  $\Pi_1$  and  $\Pi_2$ . Let the cost of  $\pi$  in  $\Pi_1$  and  $\Pi_2$  be  $h'_1(s)$  and  $h'_2(s)$ , respectively. Since  $\forall a \in \mathcal{A} : \text{cost}_1(a) + \text{cost}_2(a) = \text{cost}(a)$ , it holds that  $h'_1(s) + h'_2(s) = h^*(s)$ . Let  $h_1^*(s)$  and  $h_2^*(s)$  be the optimal solution costs for  $\Pi_1$  and  $\Pi_2$ . As  $h_1$  and  $h_2$  are admissible in  $\Pi_1$  and  $\Pi_2$ ,  $h_1(s) \leq h_1^*(s) \leq h'_1(s)$  and  $h_2(s) \leq h_2^*(s) \leq h'_2(s)$ . Thus,  $h_1(s) + h_2(s) \leq h'_1(s) + h'_2(s) = h^*(s)$ , so  $h_1(s) + h_2(s)$  is admissible in  $\Pi$ .

The  $n > 2$  case follows by induction.  $\square$

The above proof does not use any property which is only included in classical planning but not in numeric planning. Therefore, the proposition of the cost-partitioning also holds for numeric planning.

**Proposition 2.** *Given an RT and a state  $s$ , the problem of computing  $h^{\text{max}}$  is NP-hard.*

*Proof.* We show this result by reduction from the minimization version of the unbounded knapsack problem (mUKP), which is proved to be NP-hard (Zukerman et al. 2001). Let

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$J$  be a set of elements, each element  $i \in J$  has a value  $w_i$  and cost  $c_i$ . The mUKP problem consists in finding how many times  $x_i \in \mathbb{N}^0$  to select each element  $i$  to minimize  $\sum_{i \in J} c_i x_i$  such that  $\sum_{i \in J} w_i x_i \geq W$ , where  $w_i, c_i, W$  are positive rational numbers. Let  $V(X)$  be the optimal cost for an mUKP with the constraint  $\sum_{i \in J} w_i x_i \geq X$ . Since  $V(0) = 0$ , we assume that  $V(X) = 0$  if  $X \leq 0$ . Then,  $V(X)$  satisfies the following recursion formula:

$$V(X) = \min_{i \in J} V(X - w_i) + c_i.$$

Since  $\forall i \in J, w_i > 0, X - w_i < X$ . Applying the formula recursively, we reaches  $X \leq 0$  in a finite number of steps, and the recursion terminates.  $V(W)$ , the optimal cost for the mUKP, is uniquely determined by the recursion.

Given an instance of the mUKP, we build an RT with no propositional variables and with one numeric variable  $v$ . For every element  $i$  we have an action  $a_i$  such that  $\text{pre}(a_i) = \emptyset$ , and  $\text{num}(a_i) = \{v \ += \ w_i\}$ ,  $\text{cost}(a_i) = c_i$ . We set  $s_I[v] = 0$  and  $G$  contains one condition  $v \geq W$ . This RT instance is obtained in  $O(|J|)$  time. We show that the solution of the mUKP is equivalent to the solution of  $h^{\text{max}}$ . By definition,

$$h^{\text{max}}(s_I) = \hat{h}(s_I, G) = \hat{h}(s_I, \{v \geq W\}) = \hat{h}(s_I, v \geq W).$$

For a rational number  $X$ ,  $\hat{h}(s_I, v \geq X) = 0$  if  $X \leq 0$  otherwise

$$\hat{h}(s_I, v \geq X) = \min_{a_i \in \text{supp}(v \geq X)} \hat{h}(s, v \geq X - w_i) + \text{cost}(a_i)$$

because all actions have no precondition. Since  $\text{num}(a_i) = \{v \ += \ w_i\}$  for each action  $a_i$  and  $\text{supp}(v \geq X) = \{a_i \mid i \in J\}$ ,

$$\hat{h}(s_I, v \geq X) = \min_{i \in J} \hat{h}(s, v \geq X - w_i) + c_i.$$

This is equivalent to  $V(W)$ , so  $h^{\text{max}}(s_I) = \hat{h}(s_I, v \geq W)$  is the optimal cost for mUKP.  $\square$

Here, we show an RT where  $h_{\text{hbd}}^{\text{LM-cut}}$  returns an inadmissible heuristic value.

**Example 3.** Let  $\Pi^{\text{RT}} = \langle \mathcal{F}_p, \mathcal{N}, \mathcal{A}, s_I, G \rangle$  be an RT with  $\mathcal{F}_p = \{p_1, p_2, p_3, p_4, p_5, g_1, g_2\}$  and  $\mathcal{N} = \{v\}$ . Let  $s_I = \{v = 0\}$ ,  $G = \{g_1, g_2\}$ , and  $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$ , where

action	pre	eff	cost
$a_1$	$\emptyset$	$p_1$	3
$a_2$	$\emptyset$	$p_2$	1
$a_3$	$\emptyset$	$p_3$	5
$a_4$	$p_1$	$p_4$	1
$a_5$	$p_2$	$p_5$	1
$a_6$	$p_3$	$v += 1$	0
$a_7$	$p_4, p_5$	$v += 1$	1
$a_8$	$p_4$	$g_1$	0
$a_9$	$p_5$	$g_2$	0
$a_{10}$	$v \geq 1$	$g_1, g_2$	0

We show a hypergraph representation of the task in Figure 3. The optimal plan is  $\langle a_3, a_6, a_{10} \rangle$  with the cost of 5.

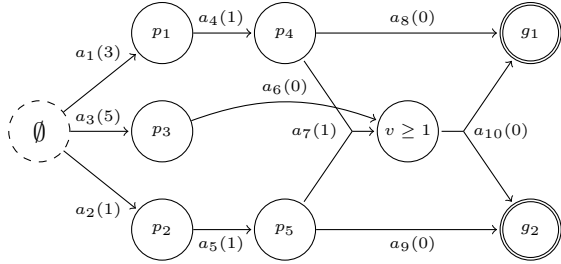


Figure 3: A hypergraph representation of Ex. 3. The action costs are shown in parentheses.

Consider  $h_{hbd}^{LM-cut}$  for this task using the justification graphs for  $h_{hbd}^{max}$ . Let  $\Pi_i^{RT}$  be the planning task with reduced action costs  $cost_i$  after extracting the  $i$ th cut  $L_i$  in the computation of  $h_{hbd}^{LM-cut}$ . For each  $\psi \in \mathcal{F}_p \cup \mathcal{F}_n$ ,  $h_{hbd}^{max}(s_I, \psi)$  in each task is as follows:

$h_{hbd}^{max}(s_I, \psi)$	$\Pi^{RT}$	$\Pi_1^{RT}$	$\Pi_2^{RT}$	$\Pi_3^{RT}$	$\Pi_4^{RT}$
$\emptyset$	0	0	0	0	0
$p_1$	3	3	0	0	0
$p_2$	1	1	1	1	0
$p_3$	5	5	5	5	5
$p_4$	4	3	0	0	0
$p_5$	2	2	2	1	0
$v \geq 1$	4	3	2	1	0
$g_1$	4	3	0	0	0
$g_2$	2	2	2	1	0

In  $\Pi^{RT}$  and  $\Pi_1^{RT}$ ,

$$pcf_{hbd}(s_I, a_6, v \geq 1) = pcf_{hbd}(s_I, a_7, v \geq 1) = p_4$$

and  $p_4 \in N^g$  since

$$\begin{aligned} \operatorname{argmin}_{a \in \operatorname{supp}(v \geq 1)} h_{hbd}^{max}(s, \operatorname{pre}(a)) &= \operatorname{argmin}_{a \in \{a_6, a_7\}} h_{hbd}^{max}(s, \operatorname{pre}(a)) \\ &= \{a_7\}, \end{aligned}$$

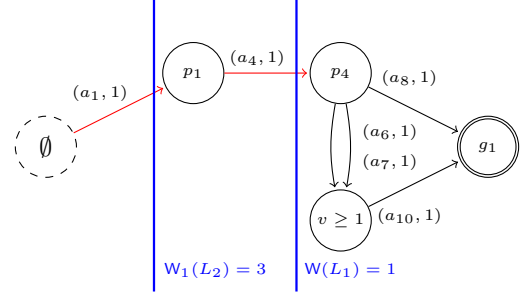
$$\operatorname{argmax}_{\psi \in \operatorname{supp}(a_7)} h_{hbd}^{max}(s, \psi) = \operatorname{argmax}_{\psi \in \{p_4, p_5\}} h_{hbd}^{max}(s, \psi) = \{p_4\},$$

and  $cost(a_7) = 0$ . Likewise,

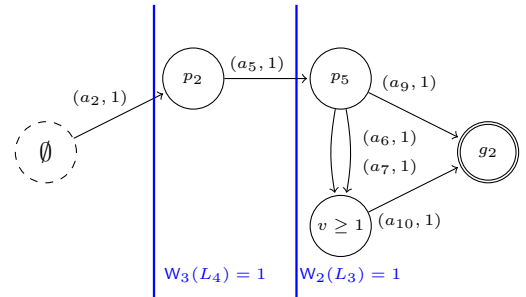
$$pcf_{hbd}(s_I, a_6, v \geq 1) = pcf_{hbd}(s_I, a_7, v \geq 1) = p_5$$

and  $p_5 \in N^g$  in  $\Pi_2^{RT}$  and  $\Pi_3^{RT}$ . The justification graphs are shown in Figure 3. Note that  $p_3$  never appears in the justification graphs because  $pcf_{hbd}(s_I, a_6, v \geq 1) \neq p_3$ . We have

$$h_{hbd}^{max}(s_I) = 4 < h^*(s_I) = 5 < h_c^{LM-cut}(s_I) = 6.$$



(a) The first and second cuts.



(b) The third and fourth cuts.

Figure 4: JGs for the RT in Ex. 3. The functions  $W$ ,  $W_1$ ,  $W_2$ , and  $W_3$  denote the cut weights of the LM-cut procedures, where action costs are reduced in each iteration.

The inadmissibility of  $h_{hbd}^{LM-cut}$  follows directly from Ex. 3.

**Proposition 3.** *The LM-cut heuristic based on the  $pcf_{hbd}$  JG,  $h_{hbd}^{LM-cut}$ , is inadmissible*

**Lemma 1.** *Assume an RT of a non-zero cost. Let  $\mathcal{G}$  be the JG corresponding to  $\Pi^{RT}$ , and let  $L$  be a directed cut in  $\mathcal{G}$  that separates  $n_\emptyset$  from  $n_g$ , such that  $W(L) > 0$ . Then,*

1.  $\partial^{in}(L)$  is a disjunctive fact landmark.
2.  $\operatorname{label}(L)$  is a disjunctive action landmark.

*Proof.* Let  $\pi$  be a plan for  $\Pi^{RT}$ . Let us construct the subsequence  $\pi'$  of the plan  $\pi$ . Let  $a_g$  be the first action in  $\pi$  that achieves the atom  $g$ . Note that by construction  $s \not\models g$ , thus such action should exist. For the action  $a_g$  we chose the first action in  $\pi$  that achieves  $pcf(a_g)$ , and repeat the process until we reach a fact  $\psi$  such that  $s \models \psi$ . Note that by construction  $\pi'$  induces a path from  $n_\emptyset$  to  $n_g$  in the JG. Thus, for every cut  $L$  that separates  $n_\emptyset$  from  $n_g$  we have that at least one fact in  $\partial^{in}(L)$  is achieved by  $\pi'$ , and  $\pi' \cap \operatorname{label}(L) \neq \emptyset$ . Thus,  $\partial^{in}(L)$  is a disjunctive fact landmark. Note that  $a_0$ , an artificial action label, is never included in  $L$ . If  $a_0 \in \operatorname{label}(L)$ ,  $\exists (n_\emptyset, n_\psi; a_0) \in L, n_\psi \in N^g$ . Since the cost of  $a_0$  is zero,  $n_\emptyset \in N^g$ , and this contradicts that

$L = (N^0, N^g \cup N^b)$ . Therefore,  $\text{label}(L)$  is a disjunctive action landmark.  $\square$

We show that the justification graphs based on  $\text{pcf}_{\text{cri}}$  and  $\text{pcf}_{\text{hbd}}$  justify  $h_{\text{cri}}^{\max}$  and  $h_{\text{hbd}}^{\max}$ , respectively.

**Proposition 4.** *Given an RT a state  $s$ , and the justification graph based on  $\text{pcf}_{\text{cri}}$ , the weight of the shortest path from  $n_\emptyset$  to  $n_\psi \in N$  is equal to  $h_{\text{cri}}^{\max}(s_I, \psi)$ .*

*Proof.* The shortest paths to other nodes can be incrementally computed in the topological order, and the first node is  $n_\emptyset$ . Therefore, we assume that the weight  $w(n_{\psi'})$  of the shortest path from  $n_\emptyset$  to  $n_{\psi'}$  is already known for all  $(n_{\psi'}, n_\psi; a) \in E$ . Assume that  $w(n_{\psi'}) = h_{\text{cri}}^{\max}(s, \psi')$  for all  $(n_{\psi'}, n_\psi; a) \in E$ . This is correct for  $\psi' = \emptyset$  because  $w(n_\emptyset) = h_{\text{cri}}^{\max}(s, \emptyset) = 0$ . Then,

$$\begin{aligned} w(n_\psi) &= \min_{(n_{\psi'}, n_\psi; a) \in E} m_a(s, \psi) \text{cost}(a) + w(n_{\psi'}) \\ &= \min_{(n_{\psi'}, n_\psi; a) \in E} m_a(s, \psi) \text{cost}(a) + h_{\text{cri}}^{\max}(s, \psi'). \end{aligned}$$

Since  $\psi' = \text{pcf}_{\text{cri}}(s, a) \in \text{argmax}_{\hat{\psi} \in \text{pre}(a)} h_{\text{cri}}^{\max}(s, \hat{\psi})$ ,

$$h_{\text{cri}}^{\max}(s, \psi') = \max_{\hat{\psi} \in \text{pre}(a)} h_{\text{cri}}^{\max}(s, \hat{\psi}) = h_{\text{cri}}^{\max}(s, \text{pre}(a)).$$

Because  $\text{supp}(\psi) = \{a \mid (n_{\psi'}, n_\psi; a) \in E\}$ ,

$$\begin{aligned} w(n_\psi) &= \min_{a \in \text{supp}(\psi)} m_a(s, \psi) \text{cost}(a) + h_{\text{cri}}^{\max}(s, \text{pre}(a)) \\ &= h_{\text{cri}}^{\max}(s, \psi). \end{aligned}$$

By mathematical induction, the shortest path from  $n_\emptyset$  to any node  $n_\psi$  is equal to  $h_{\text{cri}}^{\max}(s, \psi)$ .  $\square$

**Proposition 5.** *Given an RT a state  $s$ , and the justification graph based on  $\text{pcf}_{\text{hbd}}$ , the weight of the shortest path from  $n_\emptyset$  to  $n_\psi \in N$  is equal to  $h_{\text{hbd}}^{\max}(s_I, \psi)$ .*

*Proof.* The shortest paths to other nodes can be incrementally computed in the topological order, and the first node is  $n_\emptyset$ . Therefore, we assume that the weight  $w(n_{\psi'})$  of the shortest path from  $n_\emptyset$  to  $n_{\psi'}$  is already known for all  $(n_{\psi'}, n_\psi; a) \in E$ . Assume that  $w(n_{\psi'}) = h_{\text{hbd}}^{\max}(s, \psi')$  for all  $(n_{\psi'}, n_\psi; a) \in E$ . This is correct for  $\psi' = \emptyset$  because  $w(n_\emptyset) = h_{\text{hbd}}^{\max}(s, \emptyset) = 0$ . Then,

$$\begin{aligned} w(n_\psi) &= \min_{(n_{\psi'}, n_\psi; a) \in E} m_a(s, \psi) \text{cost}(a) + w(n_{\psi'}) \\ &= \min_{(n_{\psi'}, n_\psi; a) \in E} m_a(s, \psi) \text{cost}(a) + h_{\text{hbd}}^{\max}(s, \psi'). \end{aligned}$$

If  $\psi \in \mathcal{F}_p$ , since

$$\psi' = \text{pcf}_{\text{hbd}}(s, a, \psi) \in \text{argmax}_{\hat{\psi} \in \text{pre}(a)} h_{\text{hbd}}^{\max}(s, \hat{\psi})$$

and

$$\text{supp}(\psi) = \{a \mid (n_{\psi'}, n_\psi; a) \in E\},$$

$$h_{\text{hbd}}^{\max}(s, \psi') = \max_{\hat{\psi} \in \text{pre}(a)} h_{\text{hbd}}^{\max}(s, \hat{\psi}) = h_{\text{hbd}}^{\max}(s, \text{pre}(a)).$$

Thus,

$$\begin{aligned} w(n_\psi) &= \min_{a \in \text{supp}(\psi)} m_a(s, \psi) \text{cost}(a) + h_{\text{hbd}}^{\max}(s, \text{pre}(a)) \\ &= h_{\text{hbd}}^{\max}(s, \psi). \end{aligned}$$

Now, assume that  $\psi \in \bar{\mathcal{F}}_n$ .

$$h_{\text{hbd}}^{\max}(s, \psi') = \max_{\hat{\psi} \in \text{pre}(\hat{a})} h_{\text{hbd}}^{\max}(s, \hat{\psi})$$

where

$$\hat{a} = \text{argmin}_{a' \in \text{supp}(\psi)} h_{\text{hbd}}^{\max}(s, \text{pre}(a')).$$

Since this is independent of  $\psi'$  and  $a$ ,

$$\begin{aligned} w(n_\psi) &= \min_{a \in \text{supp}(\psi)} m_a(s, \psi) \text{cost}(a) \\ &\quad + \max_{\hat{\psi} \in \text{pre}(\hat{a})} h_{\text{hbd}}^{\max}(s, \hat{\psi}). \end{aligned}$$

As

$$\begin{aligned} &h_{\text{hbd}}^{\max}(s, \hat{\psi}) \\ &= \min_{a' \in \text{supp}(\psi)} \max_{\hat{\psi} \in \text{pre}(a')} h_{\text{hbd}}^{\max}(s, \hat{\psi}) \\ &= \min_{a' \in \text{supp}(\psi)} h_{\text{hbd}}^{\max}(s, \text{pre}(a')), \end{aligned}$$

$$\begin{aligned} w(n_\psi) &= \min_{a \in \text{supp}(\psi)} m_a(s, \psi) \text{cost}(a) \\ &\quad + \min_{a' \in \text{supp}(\psi)} h_{\text{hbd}}^{\max}(s, \text{pre}(a')) \\ &= h_{\text{hbd}}^{\max}(s, \psi) \end{aligned}$$

By mathematical induction, the shortest path from  $n_\emptyset$  to any node  $n_\psi$  is equal to  $h_{\text{hbd}}^{\max}(s, \psi)$ .  $\square$

## References

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