Extracting and Exploiting Bounds of Numeric Variables for Optimal Linear Numeric Planning – Supplementary Materials

Ryo Kuroiwa[a;*], Alexander Shleyfman[b] and J. Christopher Beck[a]

[a]Department of Mechanical and Industrial Engineering, University of Toronto
[b]Department of Computer Science, Bar-Ilan University
ORCiD ID: Ryo Kuroiwa https://orcid.org/0000-0002-3753-1644,
Alexander Shleyfman https://orcid.org/0000-0001-9187-2354,
J. Christopher Beck https://orcid.org/0000-0002-4656-8908

PICKUP domain

In this problem, n customers and one depot are given. A worker must pick up a commodity from each customer, and it can carry at most C commodities at a time. The number of commodities carried by the worker is represented by a numeric variable x. At the depot, there is a truck with a capacity Q to deliver commodities to a center. The number of commodities loaded into the truck is represented by y, and the number of commodities delivered to the center is represented by z. In the initial state, $x = y = z = 0$. The goal is to deliver all commodities to the center, i.e., $z \geq n$. The worker can load all commodities to the truck ($y \mathrel{+}= x$ and $x := 0$) if $y + x \leq Q$ at the depot. If $y + x > Q$, the worker can load commodities as much as possible ($y := Q$ and $x \mathrel{+}= y - Q$). The commodities are delivered to the center by driving the truck ($z \mathrel{+}= y$ and $y := 0$), and the truck returns to the depot after delivery.

We show a linear numeric planning task of this domain with $n = 2$ customers, the worker capacity $C = 1$, and the truck capacity $Q = 2$. The set of propositions is $\mathcal{F} = \{l_0, l_1, l_2, p_1, p_2\}$, where l_0 represents that the worker is at the depot, l_1 (l_2) represent that the worker is at customer 1 (2), and p_1 (p_2) represent that the worker does not pick up the commodity from customer 1 (2). The set of numeric variables is $\mathcal{N} = \{x, y, z\}$, the initial state is s^0 with $s^0_p = \{l_0, p_1, p_2\}$ and $s^0[x] = s^0[y] = s^0[z] = 0$, and the goal condition is $G = \{z \geq 2\}$. The set of actions is $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$, where a_1 (a_2) moves the worker to customer 1 from the depot (customer 2) and picks up the commodity, a_3 (a_4) moves the worker to customer 2 from the depot (customer 1) and picks up the commodity, a_5 (a_6) moves the worker to the depot from customer 1 (2), a_7 loads all commodities to the truck, a_8 loads as much as possible, and a_9 delivers the commodities to the center. The actions are defined in Table 1. The optimal plan is $\langle a_1, a_5, a_7, a_3, a_6, a_7, a_9 \rangle$ with the cost of 49.

When applying a_1, a_2, a_3, or a_4, $x \leq 0$, so $x \leq 1$ after the application. For a_7, the effect on x is $x := 0$. Therefore, $0 \leq x \leq 1$. When applying a_7, $y \leq 1$, and the effect on y is overestimated by $y \mathrel{+}= 1$. The effect of a_7 on y is $y := 2$, and the effect of a_8 on y is $y := 0$. Thus, $0 \leq y \leq 2$, the effect of a_9 on z is overestimated by $z \mathrel{+}= 2$.

action	pre	add	del	num	cost
a_1	$l_0, p_1, -x \geq 0$	l_1	l_0, p_1	$x \mathrel{+}= 1$	3
a_2	$l_2, p_1, -x \geq 0$	l_1	l_2, p_1	$x \mathrel{+}= 1$	5
a_3	$l_0, p_2, -x \geq 0$	l_2	l_0, p_2	$x \mathrel{+}= 1$	4
a_4	$l_1, p_2, -x \geq 0$	l_2	l_1, p_2	$x \mathrel{+}= 1$	5
a_5	l_1	l_0	l_1		3
a_6	l_2	l_0	l_2		4
a_7	$l_0, -x - y \geq -2$			$x \mathrel{+}= -x$ $y \mathrel{+}= x$	5
a_8	$l_0, x + y \geq 3$			$x \mathrel{+}= y - 2$ $y \mathrel{+}= -y + 2$	5
a_9	$l_0, y \geq 1$			$z \mathrel{+}= y$ $y \mathrel{+}= -y$	25

Table 1. Actions in the example instance of PICKUP.

We generate 20 instances with $n = 13, 14, 15, 16, 17$, $Q = \lceil n/2 \rceil, \lceil n/3 \rceil$, and $C = Q, \lceil Q/2 \rceil$. Coordinates of customers and the depot are generated uniformly at random in a 1000×1000 Euclidean space, and visiting a customer from the depot or another customer incurs the travel cost of the Euclidean distance rounded up to an integer. Loading commodities into the truck incurs the cost of $\lceil 1000\sqrt{2} \rceil$ (the maximum possible traveling cost), and driving the truck to the center incurs the cost of $\lceil 5000\sqrt{2} \rceil$ (five times the loading cost).

Extracting Bounds in Numeric Planning

Here, we describe the detailed technical proofs for the bound extraction method.

Bounds in Linear Numeric Planning

We start with setting the upper bound and lower bounds on each numeric variable $v \in \mathcal{N}$ to be $\overline{v}^0 := \infty$ and $\underline{v}^0 := -\infty$. We intend to update these bounds iteratively. Note that here we assume that $\infty + c = \infty$ and $-\infty + c = -\infty$, and $c \cdot \infty = \infty$ if $c > 0$, and

* Corresponding Author. Email: ryo.kuroiwa@mail.utoronto.ca.

$c \cdot (-\infty) = \infty$ if $c < 0$. The $-\infty$ behaves similarly under multiplication by a constant. Note that in what follows we do not use $-\infty + \infty$. We define the bounds of the numeric variables of the task in iteration i as a box $\mathsf{B}^i = \times_{v\in\mathcal{N}}[\underline{v}^i, \overline{v}^i]$, where $\underline{v}^i$ and $\overline{v}^i$ are an upper and a lower bound on v at iteration i.[1]

Let $\overline{v}_a$ and $\underline{v}_a$ be upper and lower bounds on the domain of v where the action a can be applied. For example, in the case discussed by Coles et al. [1], the action a with precondition $\mathsf{pre}(a) = \{v \geq 1\}$ has the bounds $\underline{v}_a = 1$ and $\overline{v}_a = \infty$. Since our bounds computed iteratively we denote them by $\underline{v}^i_a$ and $\overline{v}^i_a$ for each iteration $i \in \mathbb{N}$. Moreover, we define $\mathsf{preB}^i_a := \times_{v\in\mathcal{N}}[\underline{v}^i_a, \overline{v}^i_a]$ to be the $|\mathcal{N}|$-dimensional box that results from application of a to B^{i-1}.

We aim to derive tighter bounds on $\overline{v}_a$ and $\underline{v}_a$, using the linear preconditions of a. Let $\psi \in \mathsf{pre}(a)$ be a numeric precondition of the form $\sum_{v\in\mathcal{N}} w^\psi_v v \geq w^\psi_0$. Let $u \in \mathcal{N}$ be a numeric variable s.t. $w^\psi_u \neq 0$. For a to be applicable, the following condition on u must hold:

$$w^\psi_u u + \sum_{v\neq u: w^\psi_v \geq 0} w^\psi_v \overline{v}^{i-1}_a + \sum_{v\neq u: w^\psi_v \leq 0} w^\psi_v \underline{v}^{i-1}_a \geq w^\psi_0.$$

Thus, if the appropriate $\overline{v}^{i-1}_a$'s and $\underline{v}^{i-1}_a$'s are finite we can derive the upper or the lower bound on u, depending on the sign of w^ψ_u, i.e.,

$$b^i_{\psi,u} = \frac{\sum_{v\neq u: w^\psi_v \geq 0} w^\psi_v \overline{v}^{i-1}_a + \sum_{v\neq u: w^\psi_v \leq 0} w^\psi_v \underline{v}^{i-1}_a - w^\psi_0}{-w^\psi_u}. \quad (1)$$

To apply a, we need the bounds to hold altogether, thus

$$\overline{v}^i_a = \min\left\{\overline{v}^{i-1}, \min_{\psi\in\mathsf{pre}_n(a): w^\psi_v < 0} b^i_{\psi,v}\right\}, \quad (2)$$

$$\underline{v}^i_a = \max\left\{\underline{v}^{i-1}, \max_{\psi\in\mathsf{pre}_n(a): w^\psi_v > 0} b^i_{\psi,v}\right\}. \quad (3)$$

Thus, we recursively defined preB^i_a using preB^{i-1}_a and B^{i-1}.

We now calculate the numerical bounds for the effects of a based on preB^i_a. For the purpose of bound computation we transfer all numeric effect into their assignment form. For the general linear effect of a on $u \in \mathcal{N}$: $u \mathrel{+}= \xi \iff u := u + \xi$, where $\xi = \sum_{v\in\mathcal{N}} w^{a,u}_v v + w^{a,u}_0$. If action a does not effect the variable u – for the purpose of bound computation – we add the nominal effect $u := u$. Transforming increment effects into assignment effects may enable tighter bounds. For example, if we did not obtain any bounds on u under the application of a, i.e., $\overline{u}^i_a = \infty$ and $\underline{u}^i_a = -\infty$, the increment effect $u \mathrel{+}= -u + 2$ will result the bounds on u being ∞ and a $-\infty$. However, the assignment effect $u := 2$ will result in the upper and lower bounds of 2. Thus, for the rest of this section we replace all additive linear effects with assignment effects.

For the effect $(u := u + \xi) \in \mathsf{eff}_n(a)$, we compute the bounds

$$\overline{\xi}^i_u = \sum_{v\neq u: w^{a,u}_v > 0} w^{a,u}_v \overline{v}^i_a + \sum_{v\neq u: w^{a,u}_v < 0} w^{a,u}_v \underline{v}^i_a + w^{a,u}_0$$

$$\underline{\xi}^i_u = \sum_{v\neq u: w^{a,u}_v > 0} w^{a,u}_v \underline{v}^i_a + \sum_{v\neq u: w^{a,u}_v < 0} w^{a,u}_v \overline{v}^i_a + w^{a,u}_0.$$

For brevity we transform these into bounds on assigment effects. If $w^{a,u}_u \geq -1$, an upper and a lower bound on u made by are given by:

$$\overline{asn}^i_{a,u} = (w^{a,u}_u + 1)\overline{u}^i_a + \overline{\xi}^i_u, \quad (4)$$

$$\underline{asn}^i_{a,u} = (w^{a,u}_u + 1)\underline{u}^i_a + \underline{\xi}^i_u. \quad (5)$$

[1] Note that the intervals here belong to $[-\infty, \infty]$, which is a compactifications of $\mathbb{R}$.

If $w^{a,u}_u < -1$, $\overline{u}^i_a$ and $\underline{u}^i_a$ are swapped in the equations above.

Furthermore, if the linear formula of the effect is used in a precondition of the action, we can exploit the precondition to derive bounds of the effect. For example, the precondition $2x + 2y \leq 3$ imposes an upper bound of $\frac{3}{2}$ on u under the effect $u := x + y$.

Thus, let us assume that some $\psi \in \mathsf{pre}(a)$ is of the form $\psi : r\sum_{v\in\mathcal{N}} w^{a,u}_v v \leq w^\psi_0$, where $u := \sum_{v\in\mathcal{N}} w^{a,u}_v v + w^{a,u}_0$ is one of the effects of a. If $r > 0$, we replace the upper bound on the effect of a on u with $\overline{asn}^i_{a,u} := \min\left\{\overline{asn}^i_{a,u}, w^\psi_0/r + w^{a,u}_0\right\}$, if $r < 0$ we multiply the inequality by -1 and replace the bound $\underline{asn}^i_{a,u}$ accordingly.[2]

Lastly, we set the upper bound on the result of application of a to be $\overline{l}^i_{a,u} = \max\{s^0[u], \overline{asn}^i_{a,u}\}$, The lower bound on the application is defined in a similar fashion $\underline{l}^i_{a,u} = \min\{s^0[u], \underline{asn}^i_{a,u}\}$. Using these bounds we define the following effect box $\mathsf{effB}^i_a := \times_{v\in\mathcal{N}}[\underline{l}^i_{a,u}, \overline{l}^i_{a,u}]$. Note that for each $i \in \mathbb{N}$ and each $a \in \mathcal{A}$ it holds that $s^0 \in \mathsf{effB}^i_a$.

We define the next iteration of the bounds: $\overline{v}^i = \max_{a\in\mathcal{A}} \overline{l}^i_{a,u}$ and $\underline{v}^i = \min_{a\in\mathcal{A}} \underline{l}^i_{a,u}$. This gives us the box $\mathsf{B}^i := \times_{v\in\mathcal{N}}[\underline{v}_i, \overline{v}^i]$. Geometrically speaking, B^i is the smallest possible box s.t. $\bigcup_{a\in\mathcal{A}} \mathsf{effB}^i_a \subseteq \mathsf{B}^i$.

We first compute $\overline{v}^i_a$ and $\underline{v}^i_a$, bounds on variable v when action a is applicable. Using $\overline{v}^i_a$ and $\underline{v}^i_a$, we compute bounds on effects of a. Using bounds on effects of all actions, we compute $\overline{v}^i$ and $\underline{v}_i$, bounds on numeric variables. Then, we compute $\overline{v}^{i+1}_a$ and $\underline{v}^{i+1}_a$ using $\overline{v}^i_a$ and $\underline{v}^i_a$ and repeat the process. We sketch the algorithm we use to compute the bounds.

1. Initialize the initial upper bounds with ∞ and the lower bounds with $-\infty$.
2. For $i \in \mathbb{N}$ repeat 3–5 until there is no change or until i exceeds some given i_0.
3. Compute $\overline{v}^i_a$ and $\underline{v}^i_a$ for each variable v and action a in an arbitrary order (Equations (1) – (3)).
4. Compute $\overline{asn}^i_{a,v}$ and $\underline{asn}^i_{a,v}$ for each action a and variable v in an arbitrary order (Equations (4) and (5)).
5. Compute $\overline{v}^i$ and $\underline{v}^i$ for each $v \in \mathcal{N}$ in an arbitrary order.

Correctness of the Algorithm

A function f is called *increasing* (*decreasing*) if for all x and y s.t. $x \leq y$ one has $f(x) \leq f(y)$ ($f(x) \geq f(y)$). A linear function $\mathbb{R}^n \to \mathbb{R}, \vec{x} \to \sum^n_{j=1} c_j x_j + c_0$ is increasing in x_j if $c_j \geq 0$, and decreasing if $c_j \leq 0$. Constant functions are both increasing and decreasing. Let f_1 and f_2 be both increasing (decreasing) functions and let $g \in \{\min, \max\}$, then $g(f_1, f_2)$ is also increasing (decreasing).

We start the proof of correctness with the following lemma

Lemma 1 *For each $i \in \mathbb{N}$ it holds that $\mathsf{B}^{i+1} \subseteq \mathsf{B}^i$.*

Proof: We prove the claim by induction. Assume that for each $k < i$ it holds that $\mathsf{B}^{k+1} \subseteq \mathsf{B}^k$ and $\mathsf{preB}^{k+1}_a \subseteq \mathsf{preB}^k_a$ for each $a \in \mathcal{A}$. The base of induction is quite obvious since we initialize to

$$\mathsf{B}^0 = \mathsf{preB}^0_a = \times_{v\in\mathcal{N}}[-\infty, \infty],$$

thus $\mathsf{B}^1 \subseteq \mathsf{B}^0$ and $\mathsf{preB}^1_a \subseteq \mathsf{preB}^0_a$ for each $a \in \mathcal{A}$.

[2] Since all linear conditions in $\mathsf{pre}(a)$ are represented in their linear normal form (LNF) [2], we do at most one update per bound.

Let us show that $\mathsf{preB}_a^{i+1} \subseteq \mathsf{preB}_a^i$. We will show that $\overline{v}_a^{i+1} \leq \overline{v}_a^i$. The lower bound is obtained in practically the same manner. Recall that

$$\overline{v}_a^{i+1} = \min \left\{ \overline{v}^i, \min_{\psi \in \mathsf{pre}_n(a): w_v^\psi < 0} b_{\psi,v}^{i+1} \right\}.$$

We inspect this min\max formula element by element. By induction assumption, we have $\overline{v}^i \leq \overline{v}^{i-1}$. Note also that $b_{\psi,v}^{i+1}$ that appears in the equation can be seen as a linear function in $\overline{u}_a^i$'s, and $\underline{u}_a^i$'s where $u \in \mathcal{N} \setminus v$. Since the formula requires $w_v^\psi < 0$ we know that the multiplication constants of upper bounds, $-w_u^\psi / w_v^\psi$, are positive, and the constants of lower bounds are negative. By induction, we have that $\overline{u}_a^i \leq \overline{u}_a^{i-1}$ and $\underline{u}_a^i \geq \underline{u}_a^{i-1}$. Thus, for each $\psi \in \mathsf{pre}(a)$ s.t. $w_v^\psi < 0$ we have that $b_{\psi,v}^{i+1} \leq b_{\psi,v}^i$. Recalling that monotonicity preserved under the min function we have that $\overline{v}_a^{i+1} \leq \overline{v}_a^i$. The inequality $\underline{v}_a^{i+1} \geq \underline{v}_a^i$ is obtained in the same manner.

Our next step is to show that $\mathsf{effB}_a^{i+1} \subseteq \mathsf{effB}_a^i$. It is enough to show $\overline{l}_{a,v}^{i+1} \leq \overline{l}_{a,v}^i$, since the lower bounds are obtained in the same fashion. Recall that $\overline{l}_{a,v}^{i+1} = \max\{s^0[v], \overline{asn}_{a,v}^{i+1}\}$.

As before $\overline{asn}_{a,v}^i$ can be seen as linear functions over $\overline{u}_a^{i+1}$'s, and $\underline{u}_a^{i+1}$'s where $u \in \mathcal{N} \setminus v$. Since we already proved that $\mathsf{preB}_a^{i+1} \subseteq \mathsf{preB}_a^i$ for all a, we know that this bounds behave properly. Thus, by construction $\overline{asn}_{a,v}^i$ is an increasing function, over a shrinking domain. Thus, $\overline{asn}_{a,v}^{i+1} \leq \overline{asn}_{a,v}^i$. Since $s^0[v] \leq \overline{l}_{a,v}^i$, thus we have $\overline{l}_{a,v}^{i+1} \leq \overline{l}_{a,v}^i$. Therefore, $\mathsf{effB}_a^{i+1} \subseteq \mathsf{effB}_a^i$.

Furthermore, this observation grants us $\bigcup_{a \in \mathcal{A}} \mathsf{effB}_a^{i+1} \subseteq \bigcup_{a \in \mathcal{A}} \mathsf{effB}_a^i$. Thus,

$$\bigcup_{a \in \mathcal{A}} \mathsf{effB}_a^{i+1} \subseteq \mathsf{B}^i := \bigcap_{B \in \mathcal{B}^i} B,$$

where $\mathcal{B}^i = \{B \text{ is a box} \mid \bigcup_{a \in \mathcal{A}} \mathsf{effB}_a^i \subseteq B\}$. Since B^i is also a box, we have $\mathsf{B}^{i+1} \subseteq \mathsf{B}^i$. □

We proved that $\mathsf{B}^{i+1} \subseteq \mathsf{B}^i \subseteq \cdots \subseteq \mathsf{B}^0$, and by construction $(s^0)_n \in \mathsf{B}^i$ for all $i \in \mathbb{N}$. Thus, we can define $\mathsf{B}^* := \bigcap_{i=1}^\infty \mathsf{B}^i$, which we call the bounding of the task, and we know that $\mathsf{B}^* \neq \emptyset$. Moreover, since each sequence of bounds $\{\overline{v}^i\}_{i=1}^\infty$ is a monotonic sequence bounded by $s^0[v]$, we know that $\lim_{i \to \infty} \overline{v}^i = \overline{v}^*$. Thus, B^* forms a box. To finish the proof, we need to show that for every consequently applicable sequence of actions $\pi = \langle a_1, \ldots, a_m \rangle$ that is applied from s^0 and resulting in a state s it holds that $s_n \in \mathsf{B}^*$. Given we are interested in numeric bounds, it is enough to show this result for a relaxation where we ignore the propositional part, i.e., we set $\mathcal{F} = \emptyset$.

Lemma 2 *Let $s_n \in \mathsf{B}^*$ be a proper numeric state*[3] *s.t. $s_n \models \mathsf{pre}_n(a)$. Then, $s_n[\![a]\!] \in \mathsf{B}^*$.*

Proof: Let $\psi \in \mathsf{pre}_n(a)$. Since we know that $s_n \models \mathsf{pre}_n(a)$, we know that $\sum_{v \in \mathcal{N}} w_v^\psi s[v] \geq w_0^\psi$. We also know that $s_n \in \mathsf{B}^*$, hence $s_n \in \mathsf{B}^i$ for each $i \in \mathbb{N}$. First, we aim to prove by induction that $s_n \in \mathsf{preB}_a^i$ for each $i \in \mathbb{N}$. The basis of induction is trivial. Since B^0 is the whole space, and $s_n \models \mathsf{pre}_n(a)$ we have that $s_n \in \mathsf{preB}_a^1$. Recall that $\{\mathsf{preB}_a^i\}_{i=1}^\infty$ is a nested sequence of boxes, and assume that $s_n \in \cap_{k=1}^i \mathsf{preB}_a^k$.

Let us show that $s_n \in \mathsf{preB}_a^{i+1}$. Note that for each v it holds that $\underline{v}_a^i \leq s[v] \leq \overline{v}_a^i$ for each v. WLOG, assume that $w_v^\psi < 0$. Then,

$$s[v] \leq \frac{-1}{w_v^\psi} \left(\sum_{u \in \mathcal{N} \setminus \{v\}} w_u^\psi s[u] - w_0^\psi \right)$$
$$\leq \frac{-1}{w_v^\psi} \left(\sum_{u: w_u^\psi \geq 0} w_u^\psi \overline{u}_a^i + \sum_{u: w_u^\psi \leq 0} w_u^\psi \underline{u}_a^i - w_0^\psi \right) = b_{\psi,v}^i.$$

Since such inequality holds for every $\psi \in \mathsf{pre}(a)$ and v s.t. $w_v^\psi < 0$, we can write

$$s[v] \leq \min \left\{ \overline{v}^i, \min_{\psi \in \mathsf{pre}_n(a): w_v^\psi < 0} b_{\psi,v}^{i+1} \right\} = \overline{v}_a^{i+1}.$$

Hence, we have that $s_n \in \mathsf{preB}_a^{i+1}$, and thus in $s_n \in \mathsf{preB}_a^*$.

Let $u := \xi \in \mathsf{num}(a)$ be an assignment numeric effect,[4] and let us denote

$$\overline{\xi}^x = \sum_{v: w_v^{a,u} \geq 0} w_v^{a,u} \overline{v}_a^x + \sum_{v: w_v^{a,u} \leq 0} w_v^{a,u} \underline{v}_a^x + w_0^{a,u},$$
$$\underline{\xi}^x = \sum_{v: w_v^{a,u} \leq 0} w_v^{\xi} \overline{v}_a^x + \sum_{v: w_v^{a,u} \geq 0} w_v^{a,u} \underline{v}_a^x + w_0^{a,u}.$$

Since $\underline{v}_a^i \leq \underline{v}_a^* \leq s[v] \leq \overline{v}_a^* \leq \overline{v}_a^i$ for each $v \in \mathcal{N}$ and each $i \in \mathbb{N}$, it holds that $\underline{\xi}^i \leq \underline{\xi}^* \leq s[\xi] \leq \overline{\xi}^* \leq \overline{\xi}^i$ for each $i \in \mathbb{N}$.

If there is no $\psi \in \mathsf{pre}(a)$ of the form $\psi : r \sum_{v \in \mathcal{N}} w_v^{a,u} v \leq w_0^\psi$ we are almost done. Otherwise, assume there is such ψ, and WLOG, assume that $r > 0$. Since $s_n \models \mathsf{pre}_n(a)$ we have that $s[\xi] \leq \frac{w_0^\psi}{r} + w_0^{a,u}$. Thus,

$$\underline{u}^{i+1} \leq \underline{\xi}^i \leq s[\xi] \leq \min \left\{ \overline{\xi}^i, \frac{w_0^\psi}{r} + w_0^{a,u} \right\} \leq \overline{u}^{i+1}$$

i.e., $s[\xi] \in \mathsf{B}^{i+1}$. Since this claim holds for every i we have that $s[\xi] \in \mathsf{B}^*$. □

These two lemmas provide us with the following result.

Theorem 1 *Let Π be a linear numeric planning task, let B^* to be the bounding of Π. Let $\pi = \langle a_1, \ldots, a_m \rangle$ be a plan for Π, and let $\langle s^0, \ldots, s^m \rangle$ be the sequence of states that corresponds to π. Then, $s_n^k \in \mathsf{B}^*$ for each $k \in [m]$.*

Note that since $\{\mathsf{B}^i\}_{i=1}^\infty$ is a sequence of nested boxes s.t. $\mathsf{B}^* \subseteq \mathsf{B}^i$ for each i, instead of computing B^*, we can compute B^i for some large enough i.

Lastly, we present an unsatisfaibility criteria for the linear numeric planning task. Let A_G be the polytope defined by G_n the linear conditions given in the numeric goal part of the task.

Corollary 1 *Let Π be a linear numeric planning task, let B^* be the bounding of Π, and let A_G be its goal polytope. Then, $\mathsf{B}^* \cap A_G = \emptyset$ implies that Π is unsolvable.*

As before, since $\mathsf{B}^* \subseteq \mathsf{B}^i$ for each i, the task is unsolavble if $\mathsf{B}^i \cap A_G = \emptyset$ for some i. Note that since both B^i and A_G are represented as a set of linear inequalities, finding if $\mathsf{B}^* \cap A_G$ is empty or not, can be done in polynomial time [3].

[3] We assume that all values of s_n are finite.

[4] The additive effect $u \mathrel{+}= \xi \in \mathsf{num}(a)$ is transformed into assignment effect $u := u + \xi$. As a slight abuse of notation in the transformed effect we replace the original multiplicative constant of u in the additive effect, $w_u^{a,u}$, with $w_u^{a,u} := w_u^{a,u} + 1$ transforming the addtive effect into an assigment effect.

Full Proof of Theorem 2

Theorem 2 *Given action a with a SOSE $u \mathrel{+}= y{+}w$ and its second-order supporter a', let $\overline{inc}_{a',y}$ be an upper bound of the effect of a' on y and $\overline{y}_a$ be an upper bound of y when a is applicable. The optimal cost of the following optimization problem is a lower bound of the cost to achieve numeric condition $\psi : x^\psi \geq w_0^\psi$ from state s using only a' and a.*

$$\begin{aligned} &\min X\mathsf{cost}(a) + \frac{Y - s[y]}{\overline{inc}_{a',y}}\mathsf{cost}(a') \\ &\text{s.t. } w_0^\psi - s[x^\psi] = X(Y + w) \\ &\quad 0 \leq X, 0 \leq Y \leq \overline{y_a}. \end{aligned}$$

The optimal solution $(X, Y) = (X^*, Y^*)$ for the above problem is given as follows:

$$X^* = \frac{w_0^\psi - s[x^\psi]}{Y^* + w}.$$
$$Y^* = \min\left\{\overline{y}_a, \sqrt{\frac{(w_0^\psi - s[x^\psi])\overline{inc}_{a',y}\mathsf{cost}(a)}{\mathsf{cost}(a')}} - w\right\}$$

Proof: Note that the optimization function is increasing in X, since $\mathsf{cost}(a) \geq 0$. Thus, we can fix some large enough X_0, and solve the problem for $0 \leq X \leq X_0$, ensuring that the domain is compact, and the minimum exists.

Using Lagrange multipliers, we get the following function and KKT conditions:

$$\begin{aligned} \mathcal{L}(X, Y, \lambda, \mu) = \quad & X\mathsf{cost}(a) + \frac{Y - s[y]}{\overline{inc}_{a',y}}\mathsf{cost}(a') \\ & -\lambda(s[x^\psi] - w_0^\psi + X(Y + w)) \\ & +\mu(Y - \overline{y}_a). \end{aligned}$$

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial X} = \mathsf{cost}(a) - \lambda(Y + w) = 0. \\ &\frac{\partial \mathcal{L}}{\partial Y} = \frac{\mathsf{cost}(a')}{\overline{inc}_{a',y}} - \lambda X + \mu = 0. \\ &\frac{\partial \mathcal{L}}{\partial \lambda} = s[x^\psi] - w_0^\psi + X(Y + w) = 0. \\ &\frac{\partial \mathcal{L}}{\partial \mu} = Y - \overline{y}_a \leq 0 \\ &\mu \geq 0. \\ &\mu(Y - \overline{y}_a) = 0. \end{aligned}$$

By the last condition, $\mu = 0$ or $Y = \overline{y}_a$. If we assume $\mu = 0$, we get the same solution as

$$X^* = \frac{w_0^\psi - s[x^\psi]}{Y^* + w}, \tag{6}$$

$$Y^* = \sqrt{\frac{(w_0^\psi - s[x^\psi])w'\mathsf{cost}(a)}{\mathsf{cost}(a')}} - w. \tag{7}$$

where w' is replaced with $\overline{inc}_{a',y}$. If such a solution violates $Y - \overline{y}_a \leq 0$, i.e., $Y > \overline{y}_a$, then $Y = \overline{y}_a$ must hold instead of $\mu = 0$. Therefore, for Y, we take the minimum of the solution in Equation (7) and $\overline{y}_a$. □

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