Supplement for LM-Cut Heuristics for Optimal Linear Numeric Planning

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LM-Cut Admissibility: Proofs

Theorem 1. *Let Π_{OVC} be the OVC of a solvable LT with a non-zero optimal cost. Let L be a directed cut in a JG of the first-order delete-relaxation Π^1_{OVC}, where the set of actions in the cut is given by $\mathsf{lbl}(L) = \{a \mid (n_1, n_2; a) \in L\}$. For action a, let the minimum of multiplicators in the cut be $\mathsf{m}^L_a = \min_{(n_\psi, n_{\psi'}; a) \in L} \mathsf{m}_a(s, \psi')$ and cost_1 is defined as*

$$\mathsf{cost}_1(a) = \begin{cases} \frac{\mathsf{W}(L)}{\mathsf{m}^L_a} & \text{if } a \in \mathsf{lbl}(L) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Pi^1_{\text{OVC},1}$ be a copy of Π^1_{OVC} except that action a has cost $\mathsf{cost}_1(a)$. Then, the weight of the cut $\mathsf{W}(L) = \min_{e \in L} \mathsf{W}(e)$ is admissible for $\Pi^1_{\text{OVC},1}$.

Proof. The proof basically follows Thm. 1 by Kuroiwa et al. (2021), who proved the original result of the admissibility of LM-cut for RTs.

First, we show that at least one fact in $\partial^{\mathsf{in}}(L) = \{\psi' \mid (n_\psi, n_{\psi'}; a) \in L\}$ is achieved by any plan, i.e., $\partial^{\mathsf{in}}(L)$ is a disjunctive fact landmark. Let π be an s-plan and π' be a subsequence of π constructed by the following backtracking process. Pick some $g \subseteq G$ such that $s \not\models g$, and let a be the first action that achieves g. The fact $\mathsf{pcf}(s, a)$ is either a precondition of a or $u^\xi > 0$ with $(v \mathrel{+}= \xi) \in \mathsf{num}(a)$. In either case, $\mathsf{pcf}(s, a)$ is either satisfied by s, or must be achieved by π. We substitute g with $\mathsf{pcf}(s, a)$, and add to π' the first action in π that achieves $\mathsf{pcf}(s, a)$. Repeat the process until $s \models \mathsf{pcf}(s, a)$. By construction of the JG, π' corresponds to a path from $n_\emptyset$ to n_g, and $\pi \cap L \neq \emptyset$.

We aim to show that

$$\mathsf{W}(L) \leq \mathsf{cost}_1(\pi).$$

Since $\partial^{\mathsf{in}}(L)$ is a disjunctive fact landmark, any plan achieves at least one fact in $\partial^{\mathsf{in}}(L)$. Let ψ_0 be the first fact in $\partial^{\mathsf{in}}(L)$ achieved by an optimal plan π. Denote by $\mathsf{lbl}(L_{\psi_0}) := \{a \mid (n_\psi, n_{\psi_0}; a) \in L\}$. Say $s \not\models \psi_0$. We have two cases: ψ_0 is either a propositional fact, or a numeric one. In the first case, let us assume that $\psi_0 \in \mathcal{F}$. Then, there is an action $a' \in \mathsf{lbl}(L_{\psi_0})$ that achieves ψ_0 in a plan π. Hence,

$$\begin{aligned} \mathsf{W}(L) &= \mathsf{m}^L_{a'} \cdot \mathsf{cost}_1(a') \leq \mathsf{m}_{a'}(s, \psi_0) \cdot \mathsf{cost}_1(a') \\ &= \mathsf{cost}_1(a') \leq \mathsf{cost}_1(\pi), \end{aligned}$$

since $\mathsf{m}_{a'}(s, \psi_0) = 1$, where $\psi_0 \in \mathcal{F}$. In the second case, assume that $\psi_0 : u \geq w_0$ is a condition on a numeric variable u. Since $s \not\models \psi_0$ we have that $s[u] < w_0$. Let $\pi[u]$ be the resulting value of u after a sequence of actions π that achieves ψ_0 was applied to s.

If $\pi[u] = \infty$, then some action $a' \in L_{\psi_0}$ with a conditional effect e such that $a'^{,e,\infty} \in \mathcal{A}^1_{\text{cond}} \cap \mathsf{lbl}(L_{\psi_0})$ was applied to u along the application of π, and since $\mathsf{cost}(a') = \mathsf{cost}(a'^{,e,\infty})$ and $\mathsf{m}_{a',e,\infty}(s, \psi_0) = 1$, as in the propositional case,

$$\mathsf{W}(L) = \mathsf{m}^L_{a'} \cdot \mathsf{cost}_1(a') \leq \mathsf{cost}_1(a') \leq \mathsf{cost}_1(\pi).$$

Otherwise, assume $w_0 \leq \pi[u] < \infty$. This means that all actions that were applied to u within π lie inside the set $\mathcal{A}^1_{\text{core}} \cap \mathsf{lbl}(L_{\psi_0})$. Let X_a be the number of times action a appears in π, and let $u \mathrel{+}= c_a \in \mathsf{num}_1(a)$ be the effect of the action a on the variable u. Since, $w_0 \leq \pi[u]$ we have that

$$w_0 - s[u] \leq \pi[u] - s[u] = \sum_{a \in \pi \cap \mathsf{lbl}(L_{\psi_0})} X_a c_a.$$

The overall cost of these actions in the plan is given by

$$\sum_{a \in \pi \cap \mathsf{lbl}(L_{\psi_0})} X_a \mathsf{cost}_1(a) \leq \sum_{a \in \pi} X_a \mathsf{cost}_1(a) = \mathsf{cost}_1(\pi).$$

From all actions in $\mathsf{lbl}(L_{\psi_0})$ let us pick the one with the smallest ratio of $\frac{\mathsf{cost}_1(a)}{c_a}$. We denote the action that achieves this minimum by a_0. Using a_0, we bound the weight of the cut L as follows

$$\begin{aligned} \mathsf{W}(L) &= \mathsf{m}^L_{a_0} \cdot \mathsf{cost}_1(a_0) \leq \mathsf{m}_{a_0}(s, \psi_0) \cdot \mathsf{cost}_1(a_0) \\ &= \frac{(w_0 - s[u])}{c_{a_0}} \mathsf{cost}_1(a_0) \leq \frac{\mathsf{cost}_1(a_0)}{c_{a_0}} (\pi[u] - s[u]) \\ &= \frac{\mathsf{cost}_1(a_0)}{c_{a_0}} \sum_{a \in \pi \cap \mathsf{lbl}(L_{\psi_0})} X_a c_a \\ &\leq \sum_{a \in \pi \cap \mathsf{lbl}(L_{\psi_0})} X_a c_a \frac{\mathsf{cost}_1(a)}{c_a} \leq \sum_{a \in \pi} X_a \mathsf{cost}_1(a) \\ &= \mathsf{cost}_1(\pi). \end{aligned}$$

□

Lemma 1. *Let Π^2_{OVC} be the second-order relaxation of an* LT*, with the set of actions $\mathcal{A}^2$. Suppose that the numeric condition $v \geq w_0$ is achieved by sequence of actions π from state s, and v is changed by only simple effects and SOSE. By X_a we denote the number of times action a appears in π. Then,*

$$w_0 \leq s[v] + \sum_{\substack{a\in\pi: v \mathrel{+}= c_v^a \\ \in \mathsf{num}_1(a)}} c_v^a X_a + \sum_{\substack{a\in\pi: v \mathrel{+}= u \\ \in \mathsf{num}_2(a)}} X_a \left(s[u] + \sum_{\substack{\hat{a}\in\pi: u \mathrel{+}= c_u^{\hat{a}} \\ \in \mathsf{num}_1(\hat{a})}} c_u^{\hat{a}} X_{\hat{a}} \right). \tag{1}$$

Proof. Let us first look at the intuition for this bound. By definition of SOSE, an action a has a SOSE on v if

1. in the effect $v \mathrel{+}= u+c \in \mathsf{num}(a)$, u is a simple variable,
2. all actions that change u do not change v.

All actions that affect v not via SOSE, affect it either via a finite simple effect, or an infinite one. For all actions that affect v via SOSE, we define the set of corresponding simple variables

$$\mathcal{N}_1^v = \{u \in \mathcal{N}_1^2 \mid \exists a \in \mathcal{A}^2 : v \mathrel{+}= u \in \mathsf{num}_2(a)\},$$

where $\mathcal{A}^2$ are actions, and $\mathcal{N}_1^2$ are simple numeric variables in Π^2_{OVC}. By bullet 2, the actions in π can affect either v, or the simple variables in $\mathcal{N}_1^v$, but not both.

Our aim is to show that the bound is achieved when we first apply the actions with simple effects, and only afterwards apply the action with SOSE. We obtain this bound by removing the preconditions on all actions, and reordering the actions that affect the variables v and $\mathcal{N}_1^v$ to maximize the final value of v.

We prove the claim by induction. Let $a_1, a_2 \in \pi$, and let s' be some state. We start with the cases where the variables involved are invariant under the order of application of the actions a_1 and a_2, i.e., we say that these actions are commutative with respect to $\{v\} \cup \mathcal{N}_1^v$ if for each such variable x it holds that

$$s'[\![a_1]\!][\![a_2]\!][x] = s'[\![a_2]\!][\![a_1]\!][x].$$

Applying these actions requires their preconditions to hold in s' and in both resulting states. However, since we are interested in the upper bound, we may ignore the preconditions of the actions altogether.

1. Let a_1 and a_2 be the actions that affect the simple variables in $\mathcal{N}_1^v$. By definition of SOSE, both action does not affect v, thus

   $$s'[\![a_1]\!][\![a_2]\!][v] = s'[\![a_1]\!][v] = s'[\![a_2]\!][v] = s'[v].$$

   For each $u \in \mathcal{N}_1^v$ it holds that $u \mathrel{+}= c_1^u \in \mathsf{num}_1(a_1)$ and $u \mathrel{+}= c_2^u \in \mathsf{num}_1(a_2)$ where $c_1, c_2 \geq 0$. We do not actually keep the $c_1^u = 0$ or $c_2^u = 0$ effects, but we use this representation since it is more convenient writing the cases down. Thus, since the preconditions are ignored for each $u \in \mathcal{N}_1^v$ we have

   $$s'[\![a_1]\!][\![a_2]\!][u] = s'[u] + c_1^u + c_2^u = s'[\![a_2]\!][\![a_1]\!][u].$$

2. Let a_1 and a_2 be the actions have a SOSE on v. Combining the SOSE with constant effect we can write $v \mathrel{+}= u_1 + c_1 \in \mathsf{num}(a_1)$ and $v \mathrel{+}= u_2 + c_2 \in \mathsf{num}(a_2)$. By definition, a_1 and a_2 does not affect the variables in $\mathcal{N}_1^v$. Thus, once again the order of application does not matter since addition is commutative, hence

   $$s'[\![a_1]\!][\![a_2]\!][v] = s'[\![a_2]\!][\![a_1]\!][v] = s'[v] + s[u_1] + s'[u_2] + c_1 + c_2.$$

3. Let a_1 have the effect on v of the form $v \mathrel{+}= u_1 + c_1 \in \mathsf{num}(a_1)$, and a_2 to have simple effects on $u_2 \in \mathcal{N}_1^v \setminus \{u_1\}$. By definition of SOSE a_1 and a_2 must be commutative over $\{v\} \cup \mathcal{N}_1^v$, since they affect different variables in $\{v\} \cup \mathcal{N}_1^v$.

To finish the proof, we need to show that in terms of the bound, it is advantageous applying first all actions with simple effects and only afterward apply the actions with SOSE.

Let a_1 have an effect $v \mathrel{+}= u + c_1 \in \mathsf{num}(a_1)$, a_2 have a simple effect $u \mathrel{+}= c_2 \in \mathsf{num}_1(a_2)$, and X_1 and X_2 be the number of times these actions appear in the sequence of actions π. Note that first applying a_2 X_2 times and then applying a_1 X_1 times constitutes a lower bound on the application of the same number of actions in any other order, with respect to the value of v. Let $\hat{X}_1 + \tilde{X}_1 = X_1$ and $\hat{X}_2 + \tilde{X}_2 = X_2$, where the tilde actions applied prior to the hat actions in the following order: a_2 is applied $\tilde{X}_2$ times, then a_1 is applied $\tilde{X}_1$, then a_2 is applied $\hat{X}_2$ times, then a_1 is applied $\hat{X}_1$. Thus, the value of v after the application of these actions is

$$\begin{aligned}
&s'[v] + \tilde{X}_1((s'[u] + \tilde{X}_2 c_2) + c_1) + \\
&\hat{X}_1((s'[u] + (\tilde{X}_2 + \hat{X}_2)c_2) + c_1) = \\
&s'[v] + X_1((s'[u] + X_2 c_2) + c_1) - \tilde{X}_1 \hat{X}_2 c_2 \leq \\
&s'[v] + X_1((s'[u] + X_2 c_2) + c_1),
\end{aligned}$$

since $\tilde{X}_1, \hat{X}_2, c_2 \geq 0$. By induction we can reorder the application of action such that we first apply the actions that have constant effects on $\mathcal{N}_1^v$ and then apply all the SOSE effects on v, which results in the bound presented in the body of the lemma. □

Theorem 2. *Let Π^2_{OVC} be the second-order relaxation of an* LT*, with the set of actions $\mathcal{A}^2$. Suppose that numeric condition $v \geq w_0$ is achieved from state s, and v is changed by only simple effects and SOSE. The cost to achieve $v \geq w_0$ is bounded from below by* $\inf M_1 \cup M_2 \cup M_3$*, where*

$$\begin{aligned}
M_1 &= \{\frac{w_0 - s[v]}{c}\mathsf{cost}(a) \mid v \mathrel{+}= c \in \mathsf{num}_1(a), a \in \mathcal{A}^2\}, \\
M_2 &= \{\frac{w_0 - s[v]}{c + s[u]}\mathsf{cost}(a) \mid \\
&\qquad v \mathrel{+}= u + c \in \mathsf{num}(a), s[u] > 0, a \in \mathcal{A}^2\}, \\
M_3 &= \{\mathsf{m}^u_{\hat{a}_u,a}(s, v \geq w_0)\mathsf{cost}(\hat{a}_u) + \\
&\qquad \mathsf{m}^v_{\hat{a}_u,a}(s, v \geq w_0)\mathsf{cost}(a) \mid \\
&\qquad v \mathrel{+}= u \in \mathsf{num}_2(a), u \mathrel{+}= c \in \mathsf{num}_1(\hat{a}_u), \\
&\qquad \mathsf{m}^u_{\hat{a}_u,a}(s, v \geq w_0) > 0, a \in \mathcal{A}^2\}.
\end{aligned}$$

Proof. Let π be the sequence of actions to achieve $v \geq w_0$ from s using only simple effects and SOSE. To obtain the required lower bound we use Lemma 1 to formulate the following optimization problem:

$$\min_{X_a \geq 0 : a \in \mathcal{A}^2} f = \sum_{a \in \mathcal{A}^2} X_a \mathsf{cost}(a),$$

under the exact constraint

$$w_0 = s[v] + \sum_{\substack{a \in \pi : v += c_v^a \\ \in \mathsf{num}_1(a)}} c_v^a X_a + \qquad (\spadesuit)$$

$$\sum_{\substack{a \in \pi : v += u \\ \in \mathsf{num}_2(a)}} X_a \left(s[u] + \sum_{\substack{\hat{a} \in \pi : u += c_u^{\hat{a}} \\ \in \mathsf{num}_1(\hat{a})}} c_u^{\hat{a}} X_{\hat{a}} \right).$$

Note that this is the only constraint not of the form $X_a \geq 0$. Since we are solving an LP relaxation of the problem, we can set the constraint $\spadesuit$ to be exact. Let us also note that since $\mathsf{cost}(a) \geq 0$ for each a. We can set $X_a = 0$ for each a that does not appear in $\spadesuit$.

To solve this optimization problem analytically we use the method of Lagrange multipliers. Unfortunately, the direct application of this method would require us to go through a huge number of cases, thus, to ease the proof we divide the original optimization problem into two sub problems. First, using Lagrange multipliers, we evaluate the minimal cost of obtaining some value C_u for each simple variable $u \in \mathcal{N}_1^v$. Then, show that at most one SOSE action is enough to achieve a given numeric fact at the minimum cost. For each action with a SOSE $v \mathrel{+}= u$, we compute the minimal cost of achieving $v \geq w_0$ given a value $C_u \geq s[u]$. Then, combining these two observations, we plug C_u as a substitute variable, and once again use Lagrange multipliers to compute the cost of reaching $v \geq w_0$. Among all possible combinations to achieve $v \geq w_0$ using one simple effect, one SOSE effect using $s[u]$, and a combination of one simple effect that affects u and one SOSE effect $v \mathrel{+}= u$, we pick the one with the minimum cost.

We start with computing the cost of C_u. Let u be a simple variable in $\mathcal{N}_1^v$, and let A_1^u be the set of all delete-relaxed actions that affect u in a non-trivial fashion. Then, the minimal cost to obtain $C_u > s[u]$ is the solution to the following optimization problem

$$\min_{X_{\bar{a}} \geq 0} g_u = \sum_{\hat{a} \in A_1^u} X_{\hat{a}} \mathsf{cost}(\hat{a})$$

$$s[u] - C_u + \sum_{a \in A_1^u} c_u^{\hat{a}} X_{\hat{a}} = 0.$$

If there is $\hat{a} \in A_1^u$ such that $\mathsf{cost}(\hat{a}) = 0$, then the cost to achieve any $C_u \geq s[u]$ is zero. Otherwise, assume that $\mathsf{cost}(\hat{a}) > 0$ for each $\hat{a} \in A_1^u$. Note that g_u is a linear function with non-negative coefficients and non-negative variables, and linear constraints. This means that the function will indeed reach its minima. Using the Lagrange multiplier λ_u we obtain the function

$$\mathcal{L}_u = \sum_{\hat{a} \in A_1^u} X_{\hat{a}} \mathsf{cost}(\hat{a}) - \lambda_u (s[u] - C_u + \sum_{a \in A_1^u} c_u^{\hat{a}} X_{\hat{a}}).$$

Using partial derivatives over $X_{\hat{a}}$ for each $\hat{a} \in A_1^u$ we have

$$\frac{\partial \mathcal{L}_u}{\partial X_{\hat{a}}} = \mathsf{cost}(\hat{a}) - \lambda_u c_u^{\hat{a}} = 0 \implies \lambda_u = \frac{\mathsf{cost}(\hat{a})}{c_u^{\hat{a}}}.$$

Note that the only variable involved in these $|A_1^u|$ equations is λ_u. Thus, the only case when the function can obtain its minimum within the interior of its domain (for each $\hat{a}$ it holds that $X_{\hat{a}} > 0$) is the case when $\frac{\mathsf{cost}(\hat{a}_1)}{c_u^{\hat{a}_1}} = \frac{\mathsf{cost}(\hat{a}_2)}{c_u^{\hat{a}_2}} = r$ for any two $\hat{a}_1, \hat{a}_2 \in A_1^u$. Hence, we have that g_u is a constant function. To see this, apply this constant to the constraint on the optimization problem

$$0 = s[u] - C_u + \sum_{a \in A_1^u} c_u^{\hat{a}} X_{\hat{a}} = s[u] - C_u +$$

$$\frac{1}{r} \sum_{a \in A_1^u} c_u^{\hat{a}} \frac{\mathsf{cost}(\hat{a})}{c_u^{\hat{a}}} X_{\hat{a}} = s[u] - C_u + \frac{g_u}{r}.$$

Thus, the solution for the minimal problem is obtained on the border. Another way to look at it, is to use the theorem that any linear optimization function over a polygon, if it obtains its minimum, it obtains it on the vertices of this polygon. In the case when $\frac{\mathsf{cost}(\hat{a}_1)}{c_u^{\hat{a}_1}} \neq \frac{\mathsf{cost}(\hat{a}_2)}{c_u^{\hat{a}_2}}$ for some actions $\hat{a}_1, \hat{a}_2 \in A_1^u$, the minimum can not be obtained in the interior of the polytop, and hence, lies on the border. Thus, WLOG, assume that $X_{\hat{a}} = 0$ for all $\hat{a}$ except one that minimizes the cost to additive constant ratio above. Plugging these into the original condition we have

$$s[u] - C_u + c_u^{\hat{a}} X_{\hat{a}} = 0 \implies$$

$$\min_{X_{\bar{a}} \geq 0 : a \in A_1^u} g_u = \frac{C_u - s[u]}{c_u^{\hat{a}}} \mathsf{cost}(\hat{a}).$$

Thus, to obtain the minimum it is enough to us pick $\hat{a}_u$ with the minimal $\lambda_u = \frac{\mathsf{cost}(\hat{a}_u)}{c_u^{\hat{a}_u}}$. Thus, we know the cost to obtain C_u at the minimal cost. To remove the stacking up indices we denote effect of the minimum achieving action by $u \mathrel{+}= c_u \in \mathsf{num}_1(\hat{a}_u)$

Since all actions in π have either simple effects or SOSE on v, let us denote the set of actions having simple effects by $\mathcal{A}_1^\pi$ and the set of actions having SOSE by $\mathcal{A}_2^\pi$, respectively. Let s' be some state. Let us approximate from below the cost of getting from $s'[v]$ to the required $v \geq w_0$ using only effects on v. Assume that $s'[v] < w_0$. Using Lemma 1, and omitting the actions that does not affect v we have the condition

$$w_0 = s'[v] + \sum_{\substack{a \in \mathcal{A}_1^\pi : v += c_v^a \\ \in \mathsf{num}_1(a)}} c_v^a X_a + \sum_{\substack{a \in \mathcal{A}_2^\pi : v += u \\ \in \mathsf{num}_2(a)}} X_a s'[u].$$

with the optimization function

$$\min_{X_a \geq 0} g_v = \sum_{a \in \mathcal{A}_1^\pi \cup \mathcal{A}_2^\pi} X_a \mathsf{cost}(a).$$

Here all action have either simple effects or SOSE on v, thus we write these effects directly since the order of effect is evident from the formula.

1. $v \mathrel{+}= c \in \mathsf{num}(a)$ or $v \mathrel{+}= u + c \in \mathsf{num}(a)$ and $s'[u] \leq 0$. In this case the bound on the cost of reaching $v \geq w_0$ using a is
$$\frac{w_0 - s'[v]}{c}\mathsf{cost}(a).$$
2. In the case when $v \mathrel{+}= u + c \in \mathsf{num}(a)$ and $s'[u] > 0$, the bound is
$$\frac{w_0 - s'[v]}{c + s'[u]}\mathsf{cost}(a).$$

The key observation here, as in the case with C_u, is that no more than one action is enough to express the lower bound on the cost of reaching $v \geq w_0$ from s'. We use this observation to obtain the bound. For each state s' the bound depends at most on one simple variable u. Moreover, by definition of SOSE the actions in A_1^u do not affect v, thus after applying these actions to reach s' from s the value of v will not change, i.e., $s[v] = s'[v]$. Thus, to obtain a lower bound on reaching $v \geq w_0$ we estimate the sum of bounds of reaching $s'[u]$ from $s[u]$ and reaching $v \geq w_0$ from $s[v]$. Note that we have already estimated the case when $s'[u] = s[u]$ in bullets 1. and 2. Thus, the last case we need to cover is when $v \mathrel{+}= u + c \in \mathsf{num}(a)$, $s'[u] > s[u]$ and $c \geq 0$.

Let us denote the value of u in the state s' by C_u, i.e., $C_u := s'[u]$. We use the previous bound on C_u, to formulate the following minimization problem

$$\min_{X_a \geq 0, C_u \geq s[u]} X_a\mathsf{cost}(a) + \frac{C_u - s[u]}{c_u}\mathsf{cost}(\hat{a}_u),$$
$$\text{s.t. } w_0 - s[v] = X_a(c + C_u).$$

Assuming that $\mathsf{cost}(a), \mathsf{cost}(\hat{a}_u) > 0$, we apply the Lagrange method to solve this problem, and get the following function and its derivatives

$$\mathcal{L}(X_a, C_u, \lambda) = X_a\mathsf{cost}(a) + \frac{C_u - s[u]}{c_u}\mathsf{cost}(\hat{a}_u) -$$
$$\lambda(s[v] - w_0 + X_a(c + C_u)),$$
$$\frac{\partial \mathcal{L}_v}{\partial X_a} = \mathsf{cost}(a) - \lambda(c + C_u) = 0,$$
$$\frac{\partial \mathcal{L}_v}{\partial C_u} = \frac{\mathsf{cost}(\hat{a}_u)}{c_u} - \lambda X_a = 0,$$
$$\frac{\partial \mathcal{L}_v}{\partial \lambda} = s[v] - w_0 + X_a(c + C_u) = 0.$$

Mushing the formulas and their derivatives at zero we get

$$\frac{\mathsf{cost}(\hat{a}_u)(c + C_u)}{c_u\mathsf{cost}(a)} = X_a = \frac{w_0 - s[v]}{c + C_u},$$

Solving the quadratic equation and taking the positive solution we get

$$C_u = \sqrt{\frac{(w_0 - s[v])c_u\mathsf{cost}(a)}{\mathsf{cost}(\hat{a}_u)}} - c,$$
$$X_a = \sqrt{\frac{(w_0 - s[v])\mathsf{cost}(\hat{a}_u)}{c_u\mathsf{cost}(a)}}.$$

Here we set $\mathsf{m}^u_{\hat{a}_u,a}(s, v \geq w_0) = \frac{C_u - s[u]}{c_u}$ and $\mathsf{m}^v_{\hat{a}_u,a}(s, v \geq w_0) = X_a$. We only consider the case with $C_u > s[u]$, i.e., $\mathsf{m}^u_{\hat{a}_u,a}(s, v \geq w_0) > 0$. The cases where $C_u = s[u]$ or $C_u < 0$ are considered in bullets 1. and 2., and represented in the sets M_1 and M_2.

In the case of zero-cost actions we have the following bounds. Let a be an action with $\mathsf{cost}(a) = 0$. If a has a simple effect $v \mathrel{+}= c$ with $c > 0$, or a has an effect $v \mathrel{+}= s[u]$ with $s[u] > 0$, the cost to achieve $v \geq w_0$ is zero. Otherwise, assume $s[u] < 0$ and $c = 0$. In this case, for action a to have a positive effect on v we need to increase the value of u to be greater than zero, i.e., we need $C_u > 0$. But, since any arbitrary value of u greater than zero will do, the bound is set to the cost of reaching $C_u = 0$. By construction, to be applied action a already requires the precondition $u \geq 0$, thus when $s[u] < 0$, $c = 0$, and $\mathsf{cost}(a) = 0$ we have that

$$\mathsf{m}^u_{\hat{a}_u,a}(s, v \geq w_0) = -\frac{s[u]}{c_u}.$$

In the case when $u \in \mathcal{N}_1^v$ and $\mathsf{cost}(\hat{a}_u) = 0$, the bound on achieving $v \geq w_0$ is zero, since using $\hat{a}_u$ one can obtain an arbitrary large C_u.

Thus, minimum over all three cases grants us the lower bound on the cost of reaching $v \geq w_0$ from the state s. Note that in the case when C_u is applied to v the action a has to be applied at least once, thus we assume that in the case when $\mathsf{cost}(\hat{a}_u) = 0$ we have that $\mathsf{m}^v_{\hat{a}_u,a}(s, v \geq w_0) = 1$. Similarly, If $s[u] = 0$ and $\mathsf{cost}(a) = 0$, we use $\mathsf{m}^u_{\hat{a}_u,a}(s, v \geq w_0) = 1$ since we need to apply $\hat{a}_u$ at least once. ☐

Theorem 3. *Let Π_{OVC} be the* OVC *of a solvable* LT *with a non-zero optimal cost. Let L be a directed cut in a JG of the second-order delete-relaxation Π^2_{OVC}. For action a in L, let the minimum weight of edges including a be $\mathsf{W}^L(a) = \min_{e \in L: \exists a \in \mathsf{lbl}(e)} \mathsf{W}(e)$ and cost_1 be defined as*

$$\mathsf{cost}_1(a) = \begin{cases} \frac{\mathsf{W}(L)}{\mathsf{W}^L(a)}\mathsf{cost}(a) & \text{if } a \in \mathsf{lbl}(L) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Pi^2_{\text{OVC},1}$ be a copy of Π^2_{OVC} except that action a has cost $\mathsf{cost}_1(a)$. The weight of the cut $\mathsf{W}(L) = \min_{e \in L} \mathsf{W}(e)$ is admissible for $\Pi^2_{\text{OVC},1}$.

Proof. Following Theorem 1, we show that $\partial^{\mathsf{in}}(L)$ is the disjunctive fact landmark.

We show that the in a plan π there is a subsequence π' that corresponds a JG path from $n_\emptyset$ to n_g label-wise. Let a be the first action that achieves $g \in G$ with $s \not\models g$ in plan π. If g is a proposition or a achieves g by a non-SOSE, similarly to Theorem 1, we substitute g with $\mathsf{pcf}(s, a)$ and a with the first action that achieves $\mathsf{pcf}(s, a)$ and continue the process. Otherwise, let g be a numeric condition $v \geq w$. Suppose that a achieves g by effect $v \mathrel{+}= u + c$ where $v \mathrel{+}= u$ is a SOSE and $c \geq 0$. If $c > 0$ or $s[u] > 0$, an edge $(\mathsf{pcf}(s, a), g; a)$ exists in a JG, so we substitute g with $\mathsf{pcf}(s, a)$ and a with the first action that achieves $\mathsf{pcf}(s, a)$ to continue the process. Otherwise, $c = 0$ and $s[u] \leq 0$, so π achieves $s[u] > 0$ by some action a' with an effect

u += c' such that $c' > 0$. Fact $\mathsf{pcf}(s, a', a)$, which is a precondition of a' or a, must be achieved by π. If $s[u] = 0$ and $\mathsf{cost}(a) = 0$, $\mathsf{m}^u_{a',a}(s, v \geq w) = 1 > 0$. Otherwise, since $s[v] < w$ and $\mathsf{cost}(a) > 0$ or $c + s[u] < 0$,

$$\mathsf{m}^u_{a',a}(s, v \geq w) = 2\sqrt{\frac{(w - s[v])\mathsf{cost}(a)}{c'\mathsf{cost}(a')}} - \frac{c + s[u]}{c'} > 0.$$

Therefore, an edge $(\mathsf{pcf}(s, a', a), g; \langle a', a\rangle)$ exists in a JG. We substitute g with $\mathsf{pcf}(s, a', a)$ and a with the first action that achieves $\mathsf{pcf}(s, a', a)$. Repeating the process, π' is a path from $n_\emptyset$ to n_g, and $\pi' \cap L \neq \emptyset$.

Since $\partial^{\mathsf{in}}(L)$ is a disjunctive fact landmark, let ψ_0 be the first fact achieved by optimal plan π. If ψ_0 is achieved by a non-SOSE, the proof is the same as Theorem 1. Suppose that $\psi_0 : v \geq w_0$ is achieved by a SOSE v += ξ of action a. By Theorem 2, the cost to achieve ψ_0 in $\Pi^2_{\text{ovc},1}$ is lower bounded by $\inf M_1 \cup M_2 \cup M_3$, where cost is replaced with cost_1. Let M'_1, M'_2, and M'_3 be M_1, M_2, and M_3 computed by considering only actions included in L. The infimum $\inf M'_1 \cup M'_2 \cup M'_3$ is also a lower bound on the cost to achieve ψ_0 since ψ_0 is the first fact achieved in $\partial^{\mathsf{in}}(L)$. Since

$$\min_{(n_\psi, n_{\psi'}; a) \in L} \mathsf{m}_a(s, \psi') \cdot \mathsf{cost}_1(a) \leq \inf M'_1 \cup M'_2$$

and

$$\min_{(n_\psi, n_{\psi'}; \langle \hat{a}_u, a\rangle) \in L} \begin{Bmatrix} \mathsf{m}^u_{\hat{a}_u,a}(s, \psi') \cdot \mathsf{cost}_1(\hat{a}_u) \\ +\mathsf{m}^v_{\hat{a}_u,a}(s, \psi) \cdot \mathsf{cost}_1(a) \end{Bmatrix} \leq \inf M'_3,$$

$$\min \begin{Bmatrix} \min_{(n_\psi, n_{\psi'}; a) \in L} \mathsf{m}_a(s, \psi') \cdot \mathsf{cost}_1(a), \\ \min_{(n_\psi, n_{\psi'}; \langle \hat{a}_u, a\rangle) \in L} \begin{Bmatrix} \mathsf{m}^u_{\hat{a}_u,a}(s, \psi') \cdot \mathsf{cost}_1(\hat{a}_u) \\ +\mathsf{m}^v_{\hat{a}_u,a}(s, \psi) \cdot \mathsf{cost}_1(a) \end{Bmatrix} \end{Bmatrix}$$

is a lower bound on the optimal cost.

Finally, we show that the weight of any edge in L with the modified cost is greater than or equal to the weight of the cut $\mathsf{W}(L)$. We have two cases over the edges in L: either $(n_\psi, n_{\psi'}; \langle a\rangle) \in L$ or $(n_\psi, n_{\psi'}; \langle \hat{a}_u, a\rangle) \in L$.

By definition, for an edge $e = (n_\psi, n_{\psi'}; \langle a\rangle) \in L$ it holds that $\mathsf{W}^L(a) \leq \mathsf{W}(e)$. Thus,

$$\begin{aligned} \mathsf{m}_a(s, \psi') \cdot \mathsf{cost}_1(a) &= \mathsf{m}_a(s, \psi') \cdot \mathsf{cost}(a) \frac{\mathsf{W}(L)}{\mathsf{W}^L(a)} \\ &= \mathsf{W}(L) \frac{\mathsf{W}(e)}{\mathsf{W}^L(a)} \geq \mathsf{W}(L). \end{aligned}$$

For an edge $e = (n_\psi, n_{\psi'}; \langle \hat{a}_u, a\rangle) \in L$, since $\mathsf{W}^L(\hat{a}_u) \leq \mathsf{W}(e)$ and $\mathsf{W}^L(a) \leq \mathsf{W}(e)$,

$$\begin{aligned} &\mathsf{m}^u_{\hat{a}_u,a}(s, \psi')\mathsf{cost}_1(\hat{a}_u) + \mathsf{m}^v_{\hat{a}_u,a}(s, \psi')\mathsf{cost}_1(a) \geq \\ &\mathsf{W}(L) \frac{\mathsf{m}^u_{\hat{a}_u,a}(s, \psi')\mathsf{cost}(\hat{a}_u) + \mathsf{m}^v_{\hat{a}_u,a}(s, \psi')\mathsf{cost}(a)}{\max\{\mathsf{W}^L(\hat{a}_u), \mathsf{W}^L(a)\}} \geq \\ &\mathsf{W}(L) \frac{\mathsf{W}(e)}{\max\{\mathsf{W}^L(\hat{a}_u), \mathsf{W}^L(a)\}} \geq \mathsf{W}(L). \end{aligned}$$

□

Details for the Experimental Evaluation

For $h^{\text{LM-cut}}_1$ and $h^{\text{LM-cut}}_2$, we use redundant constraints in the same way as Scala et al. (2016): in the original task, for each action a, for each pair of numeric preconditions $\psi, \psi' \in \mathsf{pre}_n(a)$, we add redundant numeric condition $\psi + \psi' : \sum_{v \in \mathcal{N}} (w^\psi_v + w^{\psi'}_v)v \trianglerighteq w^\psi_0 + w^{\psi'}_0$ to $\mathsf{pre}_n(a)$ where $\trianglerighteq$ is $>$ if both of ψ and ψ' are strict inequalities and $\trianglerighteq$ is $\geq$ otherwise. We also add such redundant conditions to the goal conditions. Furthermore, in the relaxed task, for each conditional effect e of action a with effect condition $\mathsf{cond}(e) = \langle \emptyset, \{u^\xi > 0\}\rangle$, for each $u^\psi \trianglerighteq w^\psi_0 \in \mathsf{pre}(a)$, we introduce auxiliary variable $u^{\xi,\psi}$ and add redundant constraint $u^{\xi,\psi} > w^\psi_0$ to $\mathsf{cond}(e)$, which corresponds to $u^\xi + u^\psi > w^\psi_0$. Effects on $u^{\xi,\psi}$ are defined in the same way as other auxiliary variables introduced by OVC.

We found that A* with h^{irmax} returns sub-optimal plans for two instances of LINEAR-CAR-POLY in the experiments and suspect that there are bugs in the implementation of h^{irmax} by the original authors (Aldinger and Nebel 2017).

LIN-CAR Domains

Percassi, Scala, and Vallati (2021) proposed methods to translate PDDL+ (Fox and Long 2006) domains to PDDL 2.1 (Fox and Long 2003) domains by discretising them in the time. From the domains used in their work, we take LIN-CAR-POLY and LIN-CAR-EXP, which are translated from PDDL+ domain LIN-CAR (Fox and Long 2006) with the discretised-time interval of 1 by two different translation algorithms POLY and EXP, respectively. The other PDDL+ domains do not fit into our problem definition and cannot be handled by the LM-cut.

In LIN-CAR, there are three numeric variables a (acceleration), v (velocity), and d (distance), and the goal is to achieve $d = \hat{d}$ along with $v = 0$ and $a = 0$ where $\hat{d}$ is specified by an instance. In addition, $|a| \leq \hat{a}$ and $|v| \leq \hat{v}$ must be satisfied at any point of the plan execution where $\hat{a}$ and $\hat{v}$ are positive. Since the existing 10 instances are solved by all methods, we generate 24 new instances with parameters $\hat{d} \in \{1000, 2000\}$, $\hat{v} = 100$, and $\hat{a} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30\}$ and use the translated versions of them.

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